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DIFFERENTIAL EQUATIONS IN CONSTRUCTIVE ANALYSIS AND IN THE RECURSIVE REALIZABILITY TOPOS

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Dedicated to the memory of Errett Bishop

This paper can be read in at least three ways:

(1) As a description of the (constructive) theory of differential equations, as *provable* in Heyting's arithmetic + "Every $f: \mathbb{N} \to \mathbb{N}$ is recursive".

(2) As a description of the (constructive) theory of differential equations as *true* in Hyland's recursive realizability topos [8].

(3) As a description of the (constructive) theory of differential equations in computable analysis in which *all* assumptions are made computable, in contrast with studies of computability in ordinary analysis [12], [13], where *some* assumptions in classical theorems are taken to be computable, and some are not.

Thus our context is related to the one in [1], [2], with the important difference that we not only do *not* rely on Excluded middle, but not even on Markov's Principle (cf. Section 1 for the precise formulations). We have found that in this constructive recursive context one has the existence of approximate solutions (Section 4), and the Picard uniqueness and existence theorem for ordinary differential equations (Section 2). Also, one has the classical uniqueness and existence theorems for the wave equation (Section 5) and the heat equation (Section 6). (The Laplace equation will be discussed elsewhere.) On the other hand, the Cauchy-Peano existence theorem for differential equations is simply *refutable* in $HA + ECT_0$ (Section 3), an improvement of [1] and [2], but it does not constructively imply the Heine-Borel theorem (Section 3). (It *does* classically, cf. [15].)

1. Recursive realizability: a setting for computable analysis

Let us first recall the original definition given by Kleene in 1945. Given a natural number n and a sentence A of first-order intuitionistic arithmetic (HA for Heyting's arithmetic), one defines

n realizes A

by induction on the complexity of A

$$
\begin{array}{lll}
n\,\mathbf{r}\,A & \text{iff} & A, \text{ for atomic } A,\\[4pt]
n\,\mathbf{r}\,A(\wedge B) & \text{iff} & \pi_1(n)\,\mathbf{r}\,A \text{ and } \pi_2(n)\,\mathbf{r}\,B,\\[4pt]
n\,\mathbf{r}\,(A\vee B) & \text{iff} & \left[\begin{array}{c}\pi_1(n)=0 \text{ implies } \pi_2(n)\,\mathbf{r}\,A\\ \text{and}\\ \pi_1(n)\neq 0 \text{ implies } \pi_2(n)\,\mathbf{r}\,B\end{array}\right],\\[4pt]
n\,\mathbf{r}\,(A\to B) & \text{iff} & \text{for each } k \text{ such that } k\,\mathbf{r}\,A,\ \{n\}(k) \text{ is defined and}\\
 & & \{n\}(k)\,\mathbf{r}\,B,\\[4pt]
n\,\mathbf{r}\,\forall x\,A(x) & \text{iff} & \text{for each } k,\ \{n\}(k) \text{ is defined and } \{n\}(k)\,\mathbf{r}\,A(k),\\[4pt]
n\,\mathbf{r}\,\exists x\,A(x) & \text{iff} & \pi_2(n)\,\mathbf{r}\,A(\pi_1(n)),
\end{array}
$$

where $\{n\}(k)$ is the result of applying the partial recursive function of index n to k, and π_1, π_2 are primitive recursive coordinates of a pairing function.

Syntactically, one can think of realizability as a translation of HA into HA, assigning to each formula $A(x_1,\dots,x_n)$ of HA with free variables among $x_1,\dots,x_n$, a formula $x_0\,\mathbf{r}\,A(x_1,\dots,x_n)$ of HA with free variables among $x_0,x_1,\dots,x_n$.

No doubt the inductive clauses for implication and the quantifiers will allow only recursive functions as realizable functions $f\colon \mathbb{N}\to\mathbb{N}$. Furthermore, note that a formula $x_0\,\mathbf{r}\,A(x_1,\dots,x_n)$ is *provably equivalent to an almost negative formula* (i.e., one constructed from atomic formulae or formulae $\exists x\,(t=s)$ by means of $\wedge$, $\to$, $\forall$). Let ECT_0 (Extended Church's Thesis) denote the following schema in HA:

$$\mathrm{ECT}_0 \qquad \forall x\,(A(x)\to\exists y\,B(x,y))\to\exists z\,\forall x\,(A(x)\to\{z\}(x) \text{ defined} \wedge B(x,\{z\}(x))),$$

where A is almost negative. The following characterization of recursive realizability is well-known:

Syntactic Characterization Theorem. *For a sentence A of* HA:

(i) $\mathrm{HA}+\mathrm{ECT}_0\vdash(A\leftrightarrow\exists x\,(x\,\mathbf{r}\,A))$,

(ii) $\mathrm{HA}+\mathrm{ECT}_0\vdash A$ iff $\mathrm{HA}\vdash\exists x\,(x\,\mathbf{r}\,A)$.

Proof is by induction on the complexity of A (on the length of derivations in HA, resp.), cf. e.g. [16] for details.

Dana Scott was first to notice that realizability can also be understood in terms of truth values $[\![A]\!]=\{n\mid n\,\mathbf{r}\,A\}$. Thus one has a set $\Sigma=P(\mathbb{N})$ of truth values, and hence for each set X, a set Σ^X of predicates on X. Writing $A=(A_x\mid x\in X)$, $B=(B_x\mid x\in X)$ for elements of Σ^X, one can reformulate the definition of realizability given above as follows:

$$(A \wedge B)_x = A_x \wedge B_x = \{\langle n, m\rangle \mid n \in A_x \text{ and } m \in B_x\},$$

$$(A \vee B)_x = A_x \vee B_x = \{\langle 0, n\rangle \mid n \in A_x\} \cup \{\langle 1, m\rangle \mid m \in B_x\},$$

$$(A \to B)_x = A_x \to B_x = \{n \mid \text{if } k \in A_x, \text{ then } \{n\}(k) \text{ is defined and } \{n\}(k) \in B_x\},$$

$$\perp_x = \text{the empty set},$$

$$T_x = \mathbb{N}.$$

One then has a preorder $\vDash_x$ on Σ^X given by

$$A \vDash_x B \quad \text{iff} \quad \{(A \to B)_x \mid x \in X\} \text{ is inhabited.}$$

The Syntactic Characterization Theorem then says in particular that $(\Sigma^X, \vDash_x)$ is a Heyting pre-algebra (as a category, it has finite limits, finite colimits, and it is cartesian closed). This view of realizability was studied by Hyland [8], who constructed an elementary topos in which internal arithmetic is given by realizability. Dragalin [6] gave a similar algebraic interpretation, but not in category-theoretic terms. Hyland's construction came soon to be understood as a special case of a general topos-theoretic construction [9].

Let us say that $A \in \Sigma^X$ is *valid* iff $T \vDash_x A$. Then, following Hyland, one defines the *Effective Topos* **Eff** as a category whose *objects* are sets X equipped with an equality predicate in $\Sigma^{X \times X}$ such that

$$x = y \to y = x \qquad \text{(symmetry)},$$

$$x = y \wedge y = z \to x = z \qquad \text{(transitivity)}$$

are valid, and whose *morphisms* from $(X, =)$ to $(Y, =)$ are equivalence classes of predicates $G \in \Sigma^{X \times Y}$ such that

(i) *G is a functional relation from X to Y*, i.e. the following are valid:

$$G(x, y) \wedge x = x' \wedge y = y' \to G(x', y') \quad \textit{relational,}$$

$$G(x, y) \to x = x \wedge y = y \quad \textit{strict,}$$

$$G(x, y) \wedge G(x, y') \to y = y' \quad \textit{single-valued.}$$

$$x = x \to \exists y. G(x, y) \quad \textit{total.}$$

Here G is *equivalent* to H iff $G(x, y) \leftrightarrow H(x, y)$ is valid.

Eff is indeed a topos (cf. [9] for details). Hyland [8] proves the following

Semantic Characterization Theorem. *A sentence of* HA *is recursively realized iff it is true of the natural number object in* **Eff**.

One therefore has within **Eff** all counterexamples known in constructive recursive analysis [8], indeed, the analyis within **Eff** is Markov-Šanin constructive analysis [16, p. 991].

The definition of **Eff** just given is given in $\mathscr{S}$ets, analogously to the definition of realizability at the beginning of this section. One can, of course, work in the free topos instead of in $\mathscr{S}$ets: then the Syntactic Characterization Theorem extends to a completeness theorem for HAH + ECT_0 (HAH being the higher-order intuitionistic arithmetic). Working in $\mathscr{S}$ets, however, one has *Markov's Principle*:

$$\mathrm{MP} \qquad \forall n\,(A(n) \vee \neg A(n)) \wedge \neg\neg \exists n\, A(n) \rightarrow \exists n\, A(n)$$

valid in **Eff**. Indeed, the Syntactic Characterization Theorem extends to HA + MP, HAH + MP, resp. Markov–Šanin constructive analysis is axiomatized by HA + MP + ECT_0 [16, p. 991]. Mathematically, one needs MP to prove continuity of all functions $f: \mathbb{R} \rightarrow \mathbb{R}$. Heine–Borel Theorem is refuted already in HA + ECT_0. Thus in the following sections we talk about uniformly continuous functions (continuous functions on $[0,1]$ can be unbounded), and use the notation C^k accordingly. Notice, however, that our Theorem 3.1 does *not* depend on MP.

In particular, the natural number object $(\mathbb{N}, =)$ in **Eff** is given by the set of natural numbers and the equality predicate $[\![n = m]\!] = \{n\} \cap \{m\}$. In **Eff**, Dedekind reals and Cauchy reals are the same (Dependent Choice holds), namely the indices of recursive reals.

Although $\forall x \in \mathbb{R}\ (x = 0 \vee \neg(x = 0))$ is of course false in **Eff** (indeed, refutable in HA + ECT_0), one nevertheless has $\forall x \in \mathbb{R}. \neg\neg(x = 0 \vee \neg(x = 0))$, and the following useful:

Lemma (folklore). $\forall x, y \in \mathbb{R}\ (x < y \rightarrow \forall z \in \mathbb{R}\ (x < z \wedge z < y))$.

Remark. This is *true* for Dedekind reals. To *prove* the Lemma *in* HA + ECT_0, one constructs an algorithm ψ with values 1 or 2, defined on all (indices of Cauchy) reals x, y, z for which $x < y$ such that

$$\psi(x, y, z) = 1 \quad \text{implies} \quad z > x,$$

$$\psi(x, y, z) = 2 \quad \text{implies} \quad z < y.$$

Provability in HA + ECT_0 will be used in Theorem 3.1.

2. Lipschitz condition for the initial-value problem

We shall be concerned mostly with the initial-value problem for ordinary differential equations and its constructive solutions:

Initial-value Problem. Let $f(x, y)$ be a uniformly continuous function on a rectangle R around (a, b). Does there exist a differentiable function $y = \phi(x)$ on an interval I around a so that:

(i) $\quad (x, \phi(x)) \in R \quad$ for all $x \in I$,

(ii) $\phi'(x) = f(x, \phi(x))$ for all $x \in I$,

(iii) $\phi(a) = b$.

Such a function ϕ is said to be a *solution to the initial-value problem:*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(a) = b. \tag{1}$$

In Section 3 below we shall show that the (internal) initial-value problem as posed above, has no solution in the Effective Topos, (defined in Section 1). However, the situation is different if one assumes in addition the following *Lipschitz condition* on the function f:

There exists a constant $L > 0$ such that for every $(x, y_1), (x, y_2)$ in R:

$$|f(x, y_1) - f(x, y_2)| \leq L\,|y_1 - y_2|. \tag{2}$$

Note that f satisfies the Lipschitz condition if it has a bounded partial derivative $\partial f/\partial y$.

The Picard–Lindelöf method of successive approximations [5, pp. 11–13] then constructs the *unique* solution:

Picard–Lindelöf Existence and Uniqueness Theorem. *Let $f(x, y)$ be uniformly continuous function on the rectangle R given by $|x-a| \leq M_1$, $|y-b| \leq M_2$, such that $|f(x, y)| \leq M_2/M_1$ on R. If $f(x, y)$ satisfies the Lipschitz condition on R, then the initial-value problem* (1) *has a unique uniformly continuous solution $y = \phi(x)$ on the segment $|x-a| \leq M_1$.*

Our main interest in the method of successive approximations is due to its constructivity. [5, pp. 11–13] convinces one that it is clearly provable e.g., in intuitionistic higher-order arithmetic HAH (i.e., it holds in the free topos with the natural number object). In fact, since the reals and real functions are in the presence of ECT_0 uniquely given by recursive indices, it is actually provable in $\mathrm{HA} + \mathrm{ECT}_0$, cf. Section 1. In $\mathrm{HAH} + \mathrm{ECT}_0$, it is equivalent to the following:

Corollary 1.1. *Let $f(x, y)$ be an effectively uniformly continuous, computable function on the rectangle R of computable reals given by $|x-a| \leq M_1$, $|y-b| \leq M_2$, with $M_1, M_2 > 0$, a, b computable reals, such that $|f(x, y)| \leq M_2/M_1$ on R. If $f(x, y)$ satisfies the Lipschitz condition* (*with a computable constant L*) *on R, then the initial-value problem* (1) *has a unique, effectively uniformly continuous solution $y = \phi(x)$ on the segment of computable reals $|x-a| \leq M_1$.*

This is Theorem 12.1 in [2]. Our point is that it is a *consequence* of the Picard–Lindelöf theorem under ECT_0, rather than its *computable analogue*. In particular, it *is* the Picard–Lindelöf theorem in the Effective Topos of Section 1.

3. The Cauchy-Peano existence theorem fails

The Lipschitz condition fails for many simple functions occurring in (computing, engineering, ...) practice. For example, the initial-value problem

$$y' = y^{1/3}, \qquad y(0) = 0 \tag{3}$$

has infinitely many solutions on $[0, 1]$: for any c, $0 \leq c \leq 1$, the function ϕ_c given by:

$$\phi_c(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq c, \\ [\frac{2}{3}(x-c)]^{3/2}, & \text{if } c \leq x \leq 1 \end{cases}$$

is a solution of (3) on $[0, 1]$. The method of successive approximations described in Section 2 clearly depends on the constant in the Lipschitz condition, which fails for $f(x, y) = y^{1/3}$ on the rectangle $|x| \leq 1$, $|y| \leq 1$. In a case like this one can one find all solutions? Can one find any solutions?

The Cauch-Peano existence theorem claims (in classical logic) the existence (but not the uniqueness) of a solution to the initial value problem (1) for any (uniformly) continuous function f on a rectangle R. It follows from the Arzela-Ascoli Lemma (cf. [5, pp. 5–7]). S. Simpson has recently discovered [15] that König's Lemma lies at the heart of the proof. Both König's Lemma and its contraposite (the Fan Theorem) fail in **Eff**, so the question is raised about the status of the Cauchy-Peano existence theorem. In fact, one has:

Theorem 3.1. $\mathrm{HA} + \mathrm{ECT}_0$ *proves: "One can find a function* $f(x, y)$*, uniformly continuous in the rectangle* R: $|x| \leq 1$, $|y| \leq 1$, *such that* $|f(x, y)| \leq 1$ *on* R*, but such that for any segment* $[\alpha, \beta] \subset [-1, 1]$ *containing* 0*, there is no solution to the initial-value problem* $y' = f(x, y)$, $y(0) = 0$ *on* $[\alpha, \beta]$*."*

Proof. We will eliminate the apparent use of classical logic in [1]. Throughout the proof, we work informally in $\mathrm{HA} + \mathrm{ECT}_0$. We give (an index of) the uniformly continuous function $f(x, y)$ such that:

(i) $f(x, y) = -f(-x, y)$.

(ii) If $y(x)$ is a solution to (1) for x in the segment $[-2^{-n+1}, -2^{-n}]$ (with $n \geq 1$), then $y(-2^{-n+1}) = y(-2^{-n})$.

(iii) If $y(x)$ is a solution to (1) for x in the segment $[-2^{-n+1}, -2^{-n}]$ (with $n \geq 1$) such that $y(-2^{-n+1}) = 0$, then, if n is a recursive index and $\{n\}(n)$ is defined:

$$y(p_n) > 2^{-3(n+2)}, \quad \text{if } \{n\}(n) \text{ is even,}$$

$$y(p_n) < -2^{-3(n+2)}, \quad \text{if } \{n\}(n) \text{ is odd,}$$

where $p_n = \frac{1}{2}(-2^{-n+1} - 2^{-n})$ is the midpoint of the segment $[-2^{-n+1}, -2^{-n}]$.

No such function $f(x, y)$ permits a solution to (1) near $x = 0$ and satisfying the initial condition $y(0) = 0$. Indeed, suppose $y(x)$ is such a solution in the segment $[a, 0]$, $a < 0$. Let n_0 be a natural number such that $a < -2^{-n_0}$. By (ii), $y(-2^{-n_0}) =$

$y(-2^{-n})$ for all $n > n_0$. Now, since $y(x)$ is continuous and $y(0)=0$, we have $y(-2^{-n_0})=0$. But now let e be an index such that

$$\{e\}(n) \sim \psi(0, 2^{-3(n+2)}, y(p_n)) \tag{4}$$

(e is obtained by an appeal to the universal Turing machine).

The residue function ψ was defined in Section 1. Thus $\{e\}(n)=1$ or $\{e\}(n)=2$ for each natural number $n > n_0$, and

$$\{e\}(n)=1 \quad \text{implies} \quad y(p_n)>0,$$
$$\{e\}(n)=2 \quad \text{implies} \quad y(p_n)<2^{-3(n+2)}.$$

Hence by (iii):

$$\{e\}(n)=\begin{cases}1, & \text{if } \{n\}(n) \text{ is defined and even,}\\ 2, & \text{if } \{n\}(n) \text{ is defined and odd.}\end{cases}$$

But this would lead into a solution of the halting problem. Indeed, by introducing redundant computation steps, we can assume $e > n_0$. Now, if $\{e\}(e)=1$ (odd), then $\{e\}(e)=2$; and if $\{e\}(e)=2$ (even), then $\{e\}(e)=1$, a contradiction.

One constructs $f(x, y)$ as in [1], provided the discussion in [1] of solutions to the differential equation $y'=s(x, y)$, where $s(x, y)=9x(1-x)y^{1/3}$ can be constructivized. Indeed, if the initial condition is $y(0)=y_0$, then the solution in the segment $[0, 1]$ is

$$y(x)=\begin{cases}(x^2(3-2x)+y_0^{2/3})^{3/2}, & \text{if } y_0>0,\\ -(x^2(3-2x)+(-y_0)^{2/3})^{3/2}, & \text{if } y_0<0.\end{cases} \tag{5}$$

Observe that $s(x, y)$ satisfies the Lipschitz condition in any rectangle for which $|y| \geq r$, with r a positive rational. Thus these solutions are unique by the

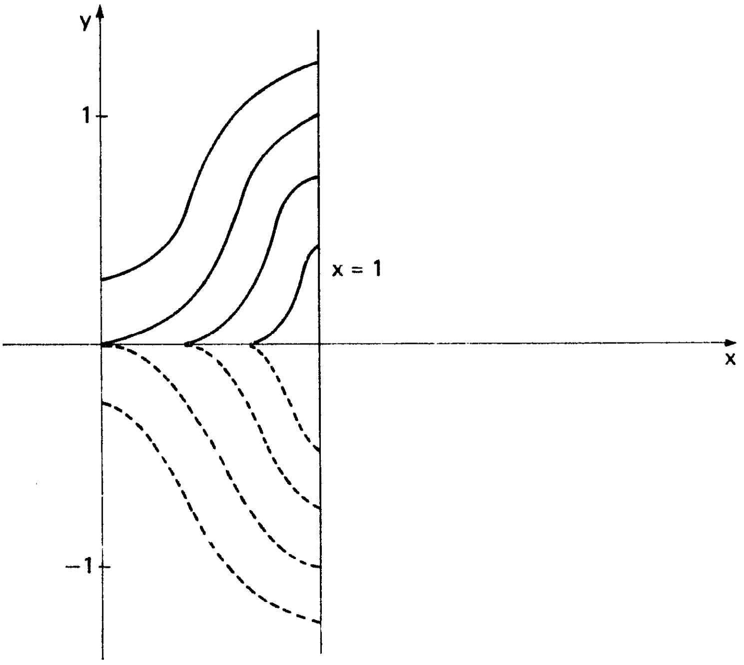

Fig. 1.

Picard-Lindelöf theorem. When $y_0=0$, there is a family of solutions in $[0,1]$:

$$y(x)=\begin{cases}0, & \text{if } 0\leq x\leq c,\\ \pm(x^2(3-2x)-c^2(3-2c))^{3/2}, & \text{if } c\leq x\leq 1,\end{cases} \tag{6}$$

where c is any real $0\leq c\leq 1$. Notice that there are actually no cases here (use $\max\{0,(x^2(3+2x)-c^2(3-2c))^{3/2}\}$ and $\min\{0,-(x^2(3-2x)-c^2(3-2c))^{3/2}\}$).

At the point $x=1$, no two of the solutions given by (5), (6) are equal. Also, there is no real which is not the value of $y(1)$ of some such solution y. Furthermore, if $y(x_0)\neq 0$ for some $x\in[0,1)$, then this solution y must increase in the absolute value in $[x_0,1]$, due to the nature of the differential equation. Now by the Picard-Lindelöf Theorem, there can be no other solution in $[x_0,1]$ with the same value at $x=1$. It follows there can be so solution in $[0,1]$ distinct from the ones given by (5), (6). Indeed, suppose $y(x)$ is such a solution. Assume

$$y(0)=0 \vee y(0)<0 \vee y(0)>0 \tag{7}$$

and

$$\exists x\,(y(x)\neq 0) \vee \forall x\,(y(x)=0). \tag{8}$$

The only case which does not immediately lead into contradiction is when $y(0)=0$ and $\exists x_0\,(y(x_0)\neq 0)$. By the above remarks, in $[x_0,1]y$ is equal to the solution given by (6), where c can be computed from $y(1)$. Now assume

$$y(c)=0 \vee y(c)>0 \vee y(c)<0. \tag{9}$$

The second and third cases immediately lead into contradiction, as does an assumption $\exists x<c.y(x)\neq 0$. But then y is equal to the solution given by (6) in the whole segment $[0,1]$, a contradiction.

We have shown that for all solutions y in $[0,1]$ distinct from those given by (5),(6), the assumptions (7), (8), (9) lead into contradiction. However, $\neg\neg(7)$, $\neg\neg(8)$, $\neg\neg(9)$ hold. Thus there are indeed no solutions on $[0,1]$ distinct from those given by (5), (6).

Remark. One could give an analogous construction in the realizability model $\mathcal{N}$ of Beeson [3, pp. 265-270], showing that Continuous Choice Principles (inconsistent with ECT_0) do not simply the Cauchy-Peano theorem. Furthermore, general considerations in [9] tell us how to build a topos in which the analysis looks like the analysis in Beeson's model $\mathcal{N}$.

S. Simpson [15] has recently shown that the Cauchy-Peano existence theorem is equivalent (in the fragment of classical second-order arithmetic in which only Σ_1^0-induction and recursive comprehension are allowed) to the compactness of $2^{\mathbb{N}}$ (the Fan Theorem), and hence to the compactness of $[0,11]$ (the Heine-Boel Theorem). On the other hand, we show:

Theorem 3.2. HAH + (Cauchy-Peano) $\nvdash$ Heine-Borel.

Proof. We utilize the sheaf model defined in [7], and discussed constructively in [14]. This proof will be entirely within HAH. Namely, one looks at the topos of sheaves over the locale obtained by adding a generic point to the locale of coperfect open sets of [0, 1]. In more detail, let $I = [0,1]$, and let $\mathcal{O}(I)$ be the locale of all open sets of [0, 1]. Let $F : \mathcal{O}(I) \to \mathcal{O}(I)$ be given by

$$F(Y) = \bigcup \{ W \in \mathcal{O}(I) \mid \exists x_1, \ldots, x_n \, [\operatorname{int}(W - \{x_1, \ldots, x_n\}) \subseteq Y] \}.$$

Let $K(I) = \{ Y \in \mathcal{O}(I) \mid F(Y) = Y \}$. Finally, let $\{*\}$ be a singleton disjoint from [0, 1], and let Ω be the set of all $Y \subseteq I \cup \{*\}$ such that $Y \cap I \in K(I)$ and $\forall t \in I \, (t \in Y \to * \in Y)$. Ω is a locale, with meets $Y \wedge W = Y \cap W$, and joins

$$\bigvee \{ Y_j \mid j \in J \} = \bigcup \{ Y_j \cap \{*\} \mid j \in J \} \cup F(\bigcup \{ Y_j \cap I \mid j \in J \}).$$

In SH(Ω), $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, and $\mathbb{R}$, $\mathbb{R}^{\mathbb{R}}$ are given by the corresponding 'old' objects in the base topos. Moreover, for any HAH formula $A(x_1, \ldots, x_n)$ with parameters $x_1, \ldots, x_n$ over $\mathbb{N}$, Q, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{R}$, $\mathbb{R}^{\mathbb{R}}$, $A(x_1, \ldots, x_n)$ holds iff $[\![A(x_1, \ldots, x_n)]\!]_{\Omega} = T$ (cf. [14]). Thus the Cauchy-Peano existence theorem gets inherited in Sh(Ω) from the base topos. On the other hand, the Heine-Borel theorem fails in Sh(Ω) (cf. [7]).

Remark. HAH + Fan Theorem $\vdash$ Heine-Borel.

4. The existence of approximate solutions

Having stated that the Lipschitz condition is often not satisfied in practice an having given a counterexample to the Cauchy-Peano existence theorem in the Effective Topos, one has to explain what *can* be done effectively. In practice one of course uses (numerical) approximation methods (Euler, Runge-Kutta, etc.). Furthermore, one sees by inspection (cf. [5, pp. 3-6]) that a nonconstructivity is introduced in a classical proof of the Cauchy-Peano existence theorem only *after* an equicontinuous sequence of polygonal approximate solutions has been constructed, to pick a uniformly converging subsequence. What one does have constructively, is a sequence of polygonal approximations.

A piecewise C^1 function ϕ on an interval I is said to be an *ε-approximate solution on I to the initial value problem* (1) if

(i) $(x, \phi(x)) \in R$ for all $x \in I$,

(ii) $|\phi'(x) - f(x, \phi(x)| < \varepsilon$ for all but finitely many $x \in I$,

(iii) $\phi(a) = b$.

Theorem 4.1 (HAH). *Let f be uniformly continuous on the rectangle R around (a, b), given by $|x-a| \le M_1$, $|y-b| \le M_2$, where $|f(x,y)| \le M_2/M_1$ on R. Then for* *any* $\varepsilon > 0$, *one can construct an* ε*-approximate solution to the initial-value problem* (1) *on the segment* $|x-a| \leq M_1$.

Proof is by the standard Eulerian polygonal approximation (cf. e.g. [5, pp. 4–5]), where the vertices $(x_0, y_0) = (a, b)$, $(x_1, y_1), \dots, (x_n, y_n)$ satisfy the difference equation

$$y_k - y_{k-1} = (x_k - x_{k-1}) \cdot f(x_{k-1}, y_{k-1}),$$

where $k = 1, \dots, n$.

Corollary 4.2. *Let* $f(x, y)$ *be an effectively uniformly continuous, computable function on the rectangle* R *of computable reals given by* $|x-a| \leq M_1$, $|y-b| \leq M_2$, *with* $M_1, M_2 > 0$, a, b *computable reals, such that* $|f(x, y)| \leq M_2/M_1$ *on* R. *Then for any* n, *one can construct a computable, effectively piecewise* C^1, 2^{-n}*-approximate solution to the initial problem* (1) *on the segment* $|x-a| \leq M_1$ *of computable reals.*

5. The wave equation

In this section we consider the existence of a solution $u(x_1, x_2, x_3, t)$ to the equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_x^2} + \frac{\partial^2 u}{\partial x_3^2} = \frac{\partial^2 u}{\partial t^2} \tag{10}$$

satisfying the initial conditions

$$u(x_1, x_2, x_3, 0) = f(x_1, x_2, x_3), \tag{11}$$

$$\frac{\partial u}{\partial t}(x_1, x_2, x_3, 0) = g(x_1, x_2, x_3). \tag{12}$$

We shall see that the situation in **Eff** is the same as the classical one (in *Sets*).

Let u_ϕ be a solution to the equation (10) subject to (11) and (12) for $f = 0$, $g = \phi$. Then the function

$$v(x_1, x_2, x_3, t) = \frac{\partial u_\phi}{\partial t}$$

satisfies the initial conditions

$$v(x_1, x_2, x_3, 0) = \phi(x_1, x_2, x_3),$$

$$\frac{\partial v}{\partial t}(x_1, x_2, x_3, 0) = \frac{\partial^2 u_\phi}{\partial t^2} = \frac{\partial^2 u_\phi}{\partial x_1^2} + \frac{\partial^2 u_\phi}{\partial x_2^2} + \frac{\partial^2 u_\phi}{\partial x_3^2} = 0.$$

Hence a solution to the equation (10) satisfying (11) and (12) is given by the formula

$$u = \frac{\partial u_f}{\partial t} + u_g. \tag{13}$$

Furthermore, such solution is unique, provided $f \in C^3(G_0)$, $g \in C^2(G_0)$ for a closed, located, bounded region G_0 in space (cf. [4]): considerations in [11, §§11–13] are completely constructive.

u_ϕ is given by the Kirchoff formula [11, §12]:

$$u_\phi(x_1, x_2, x_3, t) = \frac{1}{4\pi} \int\!\!\int_{S_t(x_1, x_2, x_3)} \frac{\phi(\alpha_1, \alpha_2, \alpha_3)}{t} \, \mathrm{d}\sigma_t \tag{14}$$

where $S_t(x_1, x_2, x_3)$ is the sphere with radius t and center (x_1, x_2, x_3) in the hyperplane $t = 0$ where ϕ is given, and $\mathrm{d}\sigma_t$ is an element of the surface of that sphere. If ϕ is C^2, so is u. In particular:

$$\frac{\partial u_\phi}{\partial t} = \frac{u_\phi}{t} + \frac{1}{4\pi t} \int\!\!\int_{S_t} \left(\frac{\partial \phi}{\partial \alpha} \, \mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3 + \frac{\partial \phi}{\partial \alpha_2} \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_3 + \frac{\partial \phi}{\partial \alpha_3} \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \right). \tag{15}$$

Putting together (15) for $\phi = f$, and (14) for $\phi = g$ into (13), we see that for u to be defined, one needs only $f \in C^1$, $g \in C^0$. Then u is only a generalized solution to (10) [11, §9], i.e., a limit of a uniformly convergent sequence of solutions to (10): (13), (14), and (15) show that u is continuous in the initial conditions, and one uses the Weierstrass approximation theorem [4, p. 100] to approximate f and g on G_0 by polynomials in three variables.

Remark. Compare the situation in **Eff** with very interesting aspects of computability in *S*ets: Boykan Pour-El and Richards [13] have improved Myhill's example [10] (in *S*ets) of a recursive, continuously differentiable real function with no recursive derivative to give the initial conditions $g = 0$ and f for which (10) has no recursive solution. Their counterexample is obstructed when the semantics of recursive realizability is applied to the statement of the classical theorem (which, as we have just seen, holds in **Eff**).

6. The heat equation

We close with a brief remark on the equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \frac{\partial u}{\partial t},$$

where u satisfies the initial condition:

$$u(x_1, x_2, x_3, 0) = f(x_1, x_2, x_3).$$

Its solution is given (in *S*ets) by the convolution operator

$$u(x_1, x_2, x_3) = \int\!\!\int\!\!\int_{\mathbb{R}^3} K_t(x_1 - \alpha_1, x_2 - \alpha_2, x_3 - \alpha_3) f(\alpha_1, \alpha_2, \alpha_3) \, \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3,$$

where

$$K_t(y_1, y_2, y_3) = (4\pi t)^{-3/2} \mathrm{e}^{-(y_1^2 + y_2^2 + y_3^2) \cdot 1/4t}.$$

Notice that

$$\iiint_{\mathbb{R}^3} K_t = 1.$$

The situation in **Eff** is the same (u being given by an explicit formula).

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